CALCULUS FOR EVERYONE

EXAM 3 ANSWER KEY

CHAPTERS 15-20

1. Why are the units of acceleration ft/s^2 or m/s^2? In other words, why the odd phrase "second *squared*"?

   The units of acceleration are ft/s^2 or m/s^2. The real meaning of these units comes through when we say them in word—"feet per second per second." Feet per second is the unit for velocity, and the other "per second" tells us that the velocity is changing every second.

2. Why did we add the negative sign to Galileo's law of fall?

   We added the negative sign to Galileo's law of fall to show that a dropped object is moving downwards.

3. What does it mean to "interpret" mathematics?

   Numbers, by themselves, don't tell us much of anything. They need to be interpreted to have meaning. To understand what numbers are saying about the world we need to know the numbers, what they represent, and the context in which they are operating. For example, we wouldn't know what $d(t) = 16t^2$ means except that we were told the context for this function (it describes a situation in which an object is dropped near the earth and air resistance can be ignored). We were told that t represents the amount of time that has elapsed since the moment the object was dropped, and d represents the distance the object has fallen in that time. Once we had all this information, we could interpret the function and its output.

4. Who revived the Platonic-Pythagorean tradition during the Scientific Revolution?

Galileo revived the Platonic-Pythagorrean tradition during the Scientific Revolution.

5. What are the two main theories of contemporary physics?

The two main theories of contemporary physics are general relativity and quantum mechanics.

6. Suppose that a golf ball is launched straight up in the air, and its height above the ground is given by the following function:

$$h(t) = -16t^2 + 200t + 6$$

a. What are the functions that describe the ball's velocity and acceleration?

$$v(t) = h'(t) = -32t + 200$$
$$a(t) = v'(t) = h''(t) = -32$$

b. At what height above the ground does the ball's travel begin? What is the ball's speed at that time?

$$h(0) = -16(0)^2 + 200(0) + 6 = 6 \text{ ft}$$
$$v(0) = -32(0) + 200 = 200 \text{ ft/s}$$

c. When is the ball at its highest? What is the maximum height?

$$v = 0 = -32t + 200 \quad \Rightarrow \quad 32t = 200 \quad \Rightarrow \quad t = \frac{200}{32} = \frac{100}{16} = \frac{25}{4} = 6.25 \text{ s}$$

$$\begin{aligned} h(6.25) &= -16(6.25)^2 + 200(6.25) + 6 \\ &= -625 + 1250 + 6 \\ &= 631 \text{ ft} \end{aligned}$$

7. Find the velocity and acceleration functions for the general free fall formula for objects near the earth's surface.

$$h(t) = -16t^2 + v_0t + h_0$$

Answer:

$$h(t) = -16t^2 + v_{.0}t + h_0$$
$$v(t) = -32t + v_0$$
$$a(t) = -32$$

8. Geometrically, what is the derivative of a function?

Geometrically, the derivative of a function is the slope of that function.

9. Why do we talk about the slope*s* (plural) of a parabola?

We talk about the slopes (plural) of a parabola because a parabola curves and so its steepness is constantly changing. This means that a parabola has a different slope at any given point, and so it has many different slopes.

10. Why doesn't it make sense to talk about secant lines and tangent lines of a *straight line*?

It doesn't make sense to talk about secant lines and tangent lines of a straight line because there are no secant or tangent lines of a straight line. There are no secant lines of a straight line because no line can cut the straight line at just two points, and there are no tangent lines because there are no lines that can touch only a single point on the straight line and still have the same slope.

11. In terms of the delta notation, what is the slope of the secant line?

In terms of delta notation, the slope of the secant line is $\frac{\Delta y}{\Delta x}$.

12. Write out the symbolic definition of a derivative and label it in terms of the slope of a secant line and the slope of a tangent line.

The symbolic definition of a derivative is $y' = \frac{dy}{dx} = \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right)$

$\frac{dy}{dx}$ refers to the slope of the tangent line.

$\lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right)$ refers to the slope of the secant line as Δx approaches 0.

13. In symbols, what is the *average* slope?

In symbols, the average slope is $\frac{\Delta y}{\Delta x}$.

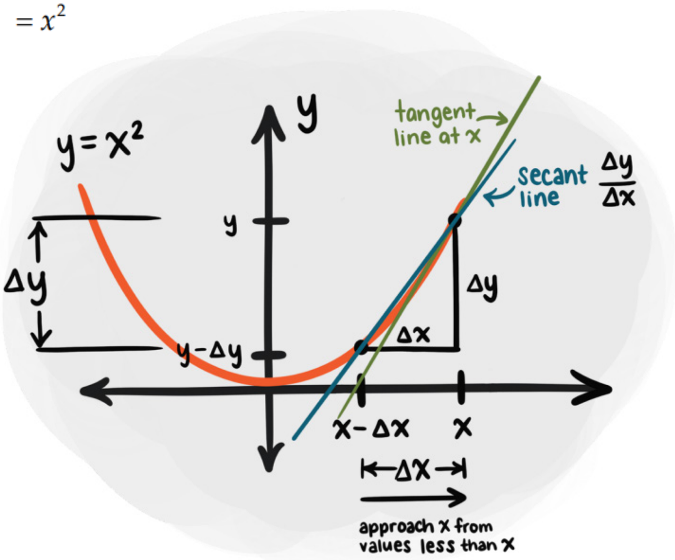
14. In symbols, what is the *instantaneous* slope? What does instantaneous mean if we're not talking about time?

In symbols, the instantaneous slope is $\frac{dy}{dx}$.

"Instantaneous" does not necessarily indicate an instant in time but the instantaneous rate of change of y with respect to x.

15. Using the Method of Increments, find the *function* that can give the slope of $y(x) = x^2$ at any x-value. Approach x from values *less than* x. Then find the slope of $y(x)$ at $x = 1$, $x = -1$, and at $x = 5$. Draw the relevant information on a Cartesian coordinate system.

$$y(x) = x^2$$



$$\begin{aligned}
\frac{\Delta y}{\Delta x} &= \frac{y(x) - y(x-\Delta x)}{\Delta x} \\
&= \frac{x^2 - (x-\Delta x)^2}{\Delta x} \\
&= \frac{x^2 - (x-\Delta x)(x-\Delta x)}{\Delta x} \\
&= \frac{x^2 - (x^2 - 2x\Delta x + (\Delta x)^2)}{\Delta x} \\
&= \frac{x^2 - x^2 + 2x\Delta x - (\Delta x)^2}{\Delta x} = \frac{2x\Delta x - (\Delta x)^2}{\Delta x} = 2x - \Delta x \qquad (\Delta x \neq 0)
\end{aligned}$$

$$\text{Slope} = y'(x)\ \frac{dy}{dx} = \lim_{\Delta x \to 0}(2x - \Delta x) = 2x$$

$$\left.\frac{dy}{dx}\right|_{x=1} = y'(2) = 2(1) = 2$$

$$\left.\frac{dy}{dx}\right|_{x=-1} = y'(-1) = 2(-1) = -2$$

$$\left.\frac{dy}{dx}\right|_{x=5} = y'(5) = 2(5) = 10$$

16. Using the Power Rule, find the *function* that can give the slope of $y(t) = -16t^2$ at any t-value. Then find the slope of $y(t)$ at $t = 0$, $t = 2$, and at $t = 8$.

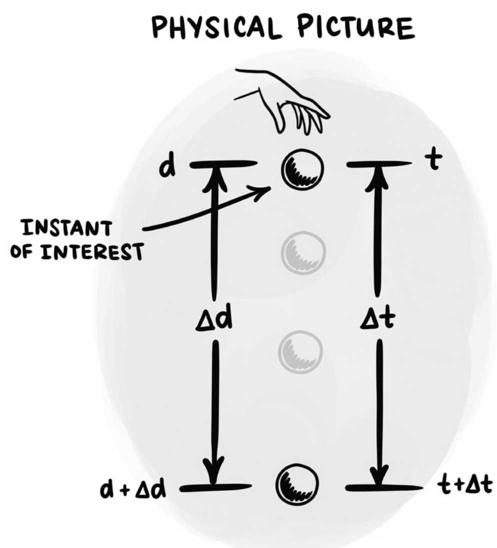
$$y(t) = -16t^2 \qquad \frac{dy}{dt} = 2\cdot(-16)t^{2-1} = -32t = y'(t)$$

$$\left.\frac{dy}{dt}\right|_{t=0} = -32(0) = 0$$

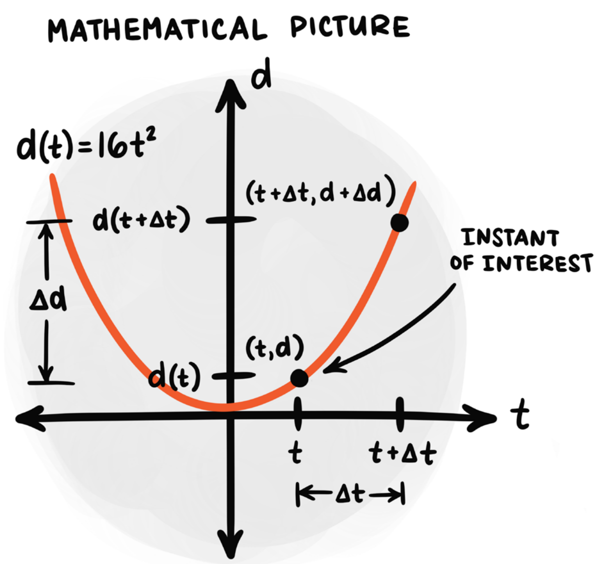
$$\left.\frac{dy}{dt}\right|_{t=2} = -32(2) = -64$$

$$\left.\frac{dy}{dt}\right|_{t=8} = -32(8) = -256$$

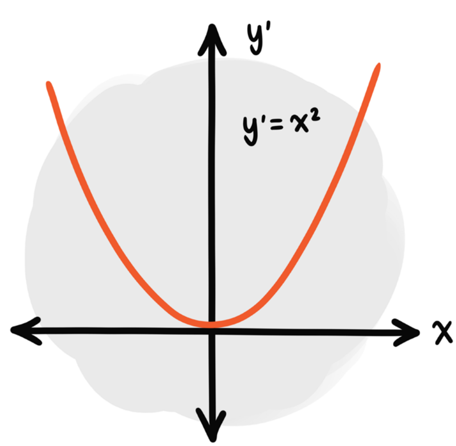
17. From memory, draw and label the *physical* picture of an object whose motion is described by Galileo's law of fall, $d(t) = 16t^2$. Use a straightedge.



18. From memory, draw and label the *mathematical* picture of an object who's motion is described by Galileo's law of fall, $d(t) = 16t^2$. Use a straightedge. Make sure to draw the line that represents the average speed between t and $t + \Delta t$, and the line representing the instantaneous speed at t.



19. Consider the following graph of $y(x)$'s derivative:



a. From just this graph, what do you know about the slope of the original function $y(x)$ for values of $x < 0$, $x = 0$, and $x > 0$?

Since the *values* of $y' = x^2$ are positive for $x < 0$, the *slope* of y (the original function) is positive for values of $x < 0$, that is, y slopes upward (when moving from left to right).

Since the *value* of $y' = x^2$ is 0 when $x = 0$, the *slope* of y (the original function) is 0 at $x = 0$, that is, y is flat exactly at $x = 0$.

Since the *values* of $y' = x^2$ are positive for $x > 0$, the *slope* of y (the original function) is positive for values of $x > 0$, that is, y slopes upward (when moving from left to right).

b. From just this graph, does the slope of the original function $y(x)$ get steeper, flatter, or stay constant for x-values greater than 0 as x moves away from 0?

As x-values move away from 0 to the right, the values of the derivative $y' = x^2$ increase, meaning that the slope of $y(x)$ gets steeper.

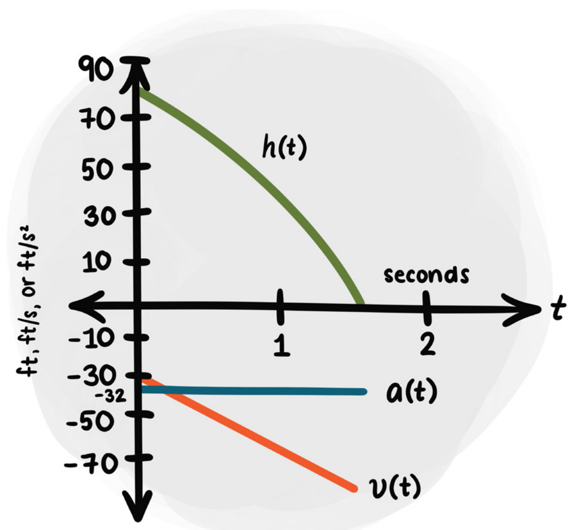
c. From just this graph, does the slope of the original function $y(x)$ get steeper, flatter, or stay constant for x-values less than 0 as x approaches 0?

As x-values approach 0 from the left, the values of the derivative $y' = x^2$ decrease, meaning that the slope of $y(x)$ gets less steep or flatter.

20. For the graph of Galileo's free fall, what happens at $t < 0$?

For the graph of Galileo's free fall, any time less than 0 is meaningless. The graph does not tell us anything about what happens before an object is dropped at $t = 0$.

21. Consider the following free fall graph (the height, velocity, and acceleration functions cross the vertical axis at 80 ft, -30 ft/s, and -32 ft/s^2, respectively):



a. What is the slope of $h(t)$ at t =0?

To find the slope of $h(t)$ we can look at $v(t)$. The value of $v(t)$ at $t = 0$ is –30. That is, $v(0) = –30$. Therefore, the slope of $h(t)$ at $t = 0$ is –30 or –30/1.

b. What is the slope of $v(t)$?

The slope of $v(t)$ is the same everywhere (that is, it's constant) and is given by the values of $a(t)$, which is –32 or –32/1.

c. What are the formulas for $h(t)$, $v(t)$, and $a(t)$?

We know that the general form of $h(t)$ is

$$h(t) = -16t^2 + v_0 t + h_0$$

From the graph we see that h_0 = 80 ft and v_0 = –30 ft/s and so

$$h(t) = -16t^2 - 30t + 80$$

So then

$$v(t) = h't = -32t - 30$$

and

$$a(t) = h''t = v'(t) = -32$$

as always.