CALCULUS FOR EVERYONE

EXAM 1 ANSWER KEY

CHAPTERS 1-8

1. There were two main ways in which Thales was important. What are they?

   The two main ways in which Thales was important are: first, he set the agenda for Western philosophy with his attempt to explain why the world around us changes while maintaining so much order; and second, he was the first mathematician (the first person to try to systematically prove mathematical statements).

2. What was the Pythagorean motto? Explain what this motto means. (Hint: How did Morris Kline unpack the motto?)

   The Pythagorean motto was "All is number." This motto has three underlying ideas: first, that the universe is ordered according to perfect mathematical laws; second, that divine reason is the orderer; and third, that human reason can discern the divine mathematical pattern.

3. What did Wigner say about the applicability of mathematics to the physical world? Why couldn't he make sense of why math "works"? How did Plato make sense of it?

   Wigner said that "The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve." In other words, he said we cannot understand why math "works." He couldn't make sense of why math "works" because he was not aware that the universe was created by an intelligent designer, a God who is a mathematician. Plato made sense of the applicability of math by saying that the world *is* mathematical, and he believed that the world was mathematical because God made it that way.

4. Explain how Plato's *Republic* is important in the history of the liberal arts. What does this have to do with Pythagoras?

   The education that Plato sketches in his book the *Republic* is where the West gets the notion of the liberal arts. There are seven liberal arts and four of them, the *quadrivium* (arithmetic, geometry, astronomy, and music) were handed down to Plato by the Pythagoreans.

5. What is Plato's challenge?

   Plato's challenge is to mathematically describe the apparently chaotic motion of the heavens.

6. What was Zeno's outrageous suggestion? What was his argument for this?

   Zeno's outrageous suggestion was that motion simply can't occur. He argued that, in order to traverse any distance, you would first have to traverse half of it. But to traverse half the distance, you would first need to traverse half of that distance (a quarter of the original distance). Zeno said that we could continue dividing distances forever, meaning that we would need to travers an infinite number of distances which would take an infinite amount of time—an eternity. Since traversing any distance would mean traversing an infinite number of distances in an infinite amount of time, Zeno concluded that motion was impossible.

7. How are mathematical lines and physical distance in space different? How are they alike? Why are their similarities important?

   Mathematical lines and physical distance in space are different because mathematical lines are mathematical objects, not physical ones, whereas physical distance is physical. We take the similarity between mathematical lines and physical distance for granted, so much so that we do not question when a line is used to represent distance. The similarity between mathematical lines and physical distance is important because it allows us to use mathematics in science.

8. What two realms does the number line unite?

   The number line unites the realm of numbers and the realm of points.

9. What is the paradox of speed? Make sure to express it in words as well as in Δ notation.

   The paradox of speed is that instantaneous speed doesn't seem to make any sense. When we consider an object changing location, it seems that it is moving at every moment in time, at every instant. However, in an instant of time, no time passes, and if no time passes, then no distance can be traveled. The paradox is that in any given instant, no motion occurs, but this does not fit with our observations. We can express this in Δ notation this way:

$$v_{ave} = \frac{\Delta x}{\Delta t} = \frac{0}{0}$$

   Here we see another problem. We have a zero in the denominator which is undefined or meaningless.

10. What do *we* say causes a dropped ball to fall? What does Aristotle say the cause is? In each case, *where* are these causes?

    We say that gravity causes a dropped ball to fall. Aristotle says that the ball's nature causes it to fall. Gravity is an external cause whereas Aristotle's answer made the cause internal.

11. What does the Greek word *physis* mean? What word do we derive from it?

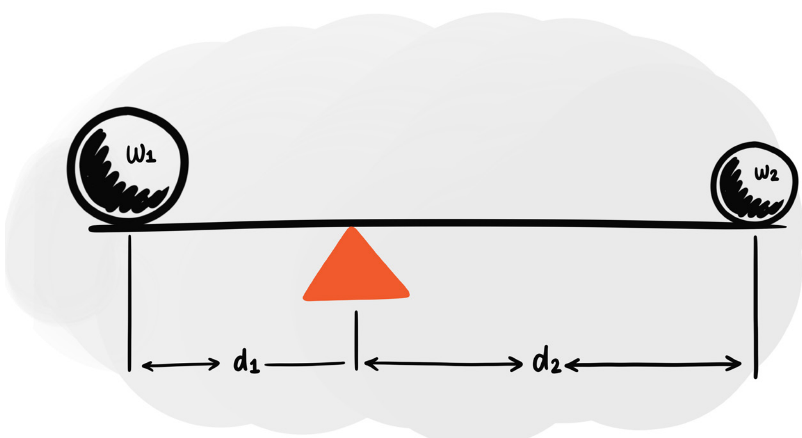
    The Greek word *physis* means nature, and it is the source of our word *physics*.

12. What is the "principle of change"?

    Aristotle said that the principle of change is an object's nature.

13. Write out, in words, Archimedes' Law of the Lever. Draw a diagram for this, label it, and write out as a formula.

    Archimedes' Law of the Lever is this: "Two magnitudes balance at distances reciprocally proportional to the magnitudes."



$$W_1 \cdot d_1 = W_2 \cdot d_2$$

$$\frac{W_1}{W_2} = \frac{d_2}{d_1}$$

14. Why is the Scientific Revolution called a "revolution"? What is it called a "scientific" revolution?

    The Scientific Revolution is called a "revolution" because it overthrew the existing regime. It is called a "scientific" revolution because it overthrew the *scientific* regime of Aristotle.

15. How might we characterize the Scientific Revolution in terms of the Platonic-Pythagorean tradition?

    We might say, as the eminent historian of science Richard Westfall did, that the Scientific Revolution was dominated or characterized by the resurgence of the Platonic-Pythagorean tradition, which looked on nature in geometric terms, convinced that the cosmos was constructed according to the principles of mathematical order.

16. In summary, what are the three main results of the Scientific Revolution? What is a fourth?

    The three main results of the Scientific Revolution are: first, the overthrow of Aristotle; second, the reintroduction of the Platonic-Pythagorean project; and third, the mathematization of motion. A fourth result of the Scientific Revolution is the development of the mechanical philosophy, the philosophy that views the universe like a machine (e.g. a clock) as opposed to Aristotle's philosophy which viewed the world more like a living organism.

17. Turn Galileo's law of fall (notice that we have a ratio)

$$\frac{d_1}{d_2} = \frac{t_1^2}{t_2^2}$$

into

$$\frac{v_{ave,1}}{v_{ave,2}} = \frac{t_1}{t_2}$$

Answer:

$$d = v_{ave} \cdot t \Rightarrow v_{ave} = \frac{d}{t} \Rightarrow \frac{d_1}{d_2} = \frac{t_1^{\ 2}}{t_2^{\ 2}} = \frac{t_1 \cdot t_1}{t_2 \cdot t_2} \Rightarrow \frac{d_1}{t_1} \cdot \frac{t_2}{d_2} = \frac{t_1}{t_2}$$

$$\Rightarrow v_{ave,1} \cdot \frac{1}{v_{ave,2}} = \frac{t_1}{t_2} \Rightarrow \frac{v_{ave,1}}{v_{ave,2}} = \frac{t_1}{t_2}$$

18. What was Galileo's view on cosmology?

Galileo had a Copernican or heliocentric view of cosmology.

19. What is the full name of Newton's *Principia* (in English)? In what year was it first published? How is unifying physics related to Newton's *Principia*? How is this related to the problem of change and calculus?

The full English name of Newton's *Principia* is *The Mathematical Principles of Natural Philosophy*. It was first published in 1687. In his work, Newton derived a set of mathematical laws that applied to the entire cosmos, in other words, he derived a set of mathematical laws that unified physics. The mathematical principles required calculus and were a major leap in solving the mathematized problem of change set by the Pythagoreans and Plato.

20. What are the three Pythagorean theses that Morris Kline said were the core beliefs underlying the Scientifict Revolution?

The three Pythagorean theses that Morris Kline said were the core beliefs underlying the Scientific Revolution are:

1. **The universe is ordered according to perfect mathematical laws**
2. **Divine reason is the orderer**
3. **Human reason can discern the divine mathematical pattern**

21. Very briefly—in a single sentence—what is calculus?

Calculus is the mathematics of change.

22. What were the four main problems that calculus was invented to solve?

The four main problems that calculus was invented to solve were:

1. **Describing continually changing velocity and acceleration**
2. **Finding the slope of a curve (including minimum and maximum values)**
3. **Finding the tangent line to a curve**
4. **Finding the area under a curve**

23. What does it mean to *quantify* a property?

To quantify a property means to represent a physical property with numbers.

24. What is a function?

A function is a relation between variables.

25. What is a graph and why is it useful?

A graph is a picture of a function, it *shows* how variables behave together. It is useful because sometimes it is much easier to *see* how variables behave than to have their behavior described in words or formulas.

26. What is analytic geometry?

Analytic geometry is a mathematical system that combines algebraic formulas with pictures in Cartesian coordinates. In other words, analytic geometry is the algebraic representation of shapes and the geometrical representation of algebraic formulas.

27. What are the three main concepts of calculus? What law tells us that all of them are related?

The three main concepts of calculus are:

1. **The limit of a function**
2. **The derivative of a function**
3. **The integral of a function**

28. Finish the sentence, "limit = the value you approach but ___________." Why does this make the "method of substitution" somewhat misleading? Make sure you know who Tantalus is.

"Limit = the value you approach but <u>never reach</u>." This makes the method of substitution somewhat misleading because this method can make it look as if we're allowing our input value to equal the forbidden value when we can really only approach the forbidden value.

29. How does the metaphor of standing versus seeing help explain why limits are exact even though they are only approached but never reached?

The metaphor of standing versus seeing helps explain why limits are exact even though they are only approached but never reached because we understand that you do not need to stand on a gap to know exactly where it is. Instead, if you can get close to the gap, you can see exactly where it is, even without standing directly on it.

30. Evaluate the following (if there is a value that the variable cannot be, make sure to note it):

a. $\lim\limits_{x \to 0} (x^2+2)$	**Answer:** $\lim\limits_{x \to 0}\left(x^2+2\right) = 0+2 = 2$

b. $\lim_{x \to 3}\left(\frac{x^2-9}{x-3}\right)$

Answer:

$$\frac{x^2-9}{x-3}=\frac{(x+3)\cancel{(x-3)}}{\cancel{x-3}}=x+3 \quad \Rightarrow \quad \lim_{x\to 3}\left[\frac{x^2-9}{x-3}\right]=\lim_{x\to 3}(x+3)=6$$

$(x \neq 3)$

c. $\lim_{x \to 4}\left(\frac{x-4}{x^2-x-12}\right)$

Answer:

$$\frac{x-4}{x^2-x-12}=\frac{\cancel{x-4}}{(\cancel{x-4})(x+3)}=\frac{1}{x+3} \quad \Rightarrow \quad \lim_{x\to 4}\left[\frac{x-4}{x^2-x-12}\right]$$

$$\Rightarrow \lim_{x\to 4}\left[\frac{1}{x+3}\right]=\frac{1}{7} \qquad (x \neq 4 \text{ or} -3)$$

d. $\lim_{x \to 0}\left(\frac{x^3}{x}\right)$

Answer:

$$\lim_{x\to 0}\left[\frac{x^3}{x}\right]=\lim_{x\to 0}\left(x^2\right)=0 \qquad (x \neq 0)$$