CALCULUS FOR EVERYONE

EXAM 2 ANSWER KEY

CHAPTERS 9-14

1. Write the definition of average speed in delta notaion.

$$v_{ave} = \frac{\Delta d}{\Delta t}$$

2. What is acceleration?

   Acceleration means that a speed is not constant, or that the speed is changing.

3. In words, describe the method of approximation for finding (i.e., approximating) instantaneous speed at some particular instant.

   Using the method of approximation, we can find the instantaneous speed at some particular instant by calculating the average speed over smaller and smaller time intervals near the particular instant. As the time interval approaches 0 seconds, the average velocity approaches the instantaneous velocity at the particular instant.

4. Define instantaneous speed/velocity in symbols.

$$v = \lim_{\Delta t \to 0} (v_{ave}) = \lim_{\Delta t \to 0} \left(\frac{\Delta d}{\Delta t} \right)$$

5. Using the method of approximation, find the instantaneous speed of a falling object after it has fallen 2 seconds. Also calculate how far it has fallen during that time.

$$v_{ave} = \frac{\Delta d}{\Delta t} = \frac{d(2)-d(1.75)}{2-1.75} = \frac{16(2)^2-16(1.75)^2}{.25} = \frac{64-49}{.25} = 60 \text{ ft/s}$$
$$v_{ave} = \frac{d(2)-d(1.9)}{2-1.9} = \frac{16(2)^2-16(1.9)^2}{.1} = \frac{64-57.6}{.1} = 62.4 \text{ ft/s}$$
$$v_{ave} = \frac{d(2)-d(1.99)}{2-1.99} = \frac{16(2)^2-16(1.99)^2}{.01} = \frac{64-63.36}{.01} = 63.84 \text{ ft/s}$$
$$v_{ave} = \frac{d(2)-d(1.999)}{2-1.999} = \frac{16(2)^2-16(1.999)^2}{.001} = \frac{64-63.936}{.001} = 63.984 \text{ ft/s}$$
$$v_{ave} = \frac{d(2)-d(1.9999)}{2-1.9999} = \frac{16(2)^2-16(1.9999)^2}{.0001} = \frac{64-63.994}{.0001} = 64.00 \text{ ft/s}$$

Approaching 64 ft/s.

$d = 16(2)^2 = 64$ ft

6. What are the two steps of the Method of Increments?

The two steps of the Method of Increments are first, find a formula for $\Delta d/\Delta t$, and second, take the limit of $\Delta d/\Delta t$ as Δt approaches 0.

7. In a single sentence, what is the general procedure for finding instantaneous speed?

The general procedure for finding instantaneous speed is to find a formula for $\Delta d/\Delta t$ where Δt is no longer in the denominator, then substitute 0 for Δt.

8. Using the Method of Increments, find the instantaneous speed of a falling object after it has fallen 2 seconds. Approach the instant from times less than 2 seconds. Compare with your answer from question 5.

$$\begin{aligned}\frac{\Delta d}{\Delta t} &= \frac{d(2)-d(2-\Delta t)}{\Delta t} = \frac{16(2)^2-16(2-\Delta t)^2}{\Delta t} = \frac{64-16(2-\Delta t)(2-\Delta t)}{\Delta t} = \\ &= \frac{64-16(4-4\Delta t+(\Delta t)^2)}{\Delta t} = \frac{64-64+64\Delta t-16(\Delta t)^2}{\Delta t} = \frac{64\Delta t-16(\Delta t)^2}{\Delta t} \\ &= 64-16\Delta t \qquad (\Delta t \neq 0) \\ v_2 &= \lim_{\Delta t \to 0}(64-16\Delta t) = 64 \text{ ft/s}\end{aligned}$$

9. Why ought we to be amazed at our ability to arrive at a method for calculating instantaneous speed? How is this related to the Platonic-Pythagorean view of the world?

We ought to be amazed at our ability to arrive at a method for calculating instantaneous speed because we arrived at a relation between instantaneous speed and time *without doing any experiments whatsoever*. This is an affirmation of the Platonic-Pythagorean view of the world that the world can be descrived mathematically because we discovered something true about the world using just mathematics.

10. What are the three central concepts of calculus? What mathematical law ties them all together? Which have we studied so far?

The three central concepts of calculus are: the limit of a function, the derivative of a function, and the integral of a function. The Fundamental Theorem of Calculus ties them all together. So far we have studied the limit of a function.

11. The free fall formula for objects near the surface of Jupiter is roughly

$$d(t) = 40t^2$$

Using the Method of Increments, find the velocity function for objects near Jupiter's surface. Approach the instant t from times less than t.

$$\begin{aligned}
\frac{\Delta d}{\Delta t} &= \frac{d(t)-d(t-\Delta t)}{\Delta t} = \frac{40(t)^2-40(t-\Delta t)^2}{\Delta t} = \frac{40t^2-40(t-\Delta t)(t-\Delta t)}{\Delta t} \\
&= \frac{40t^2-40(t^2-2t\Delta t+(\Delta t)^2)}{\Delta t} = \frac{40t^2-40t^2+80t\Delta t-40(\Delta t)^2}{\Delta t} \\
&= \frac{80t\Delta t-40(\Delta t)^2}{\Delta t} = 80t-40\Delta t \\
v(t) &= \lim_{\Delta t\to 0}(80t-40\Delta t) = 80t
\end{aligned}$$

12. Express, "The instantaneous rate of change in y with respect to x" using symbols.

$$\lim_{\Delta x\to 0}\left(\frac{\Delta y}{\Delta x}\right)$$

13. Express, "The average rate of change in y with respect to x" using symbols.

$$\frac{\Delta y}{\Delta x}$$

14. Are all derivatives limits? Are all limits derivatives?

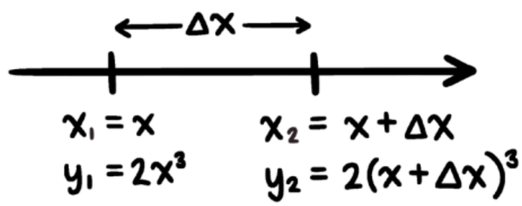
All derivatives are limits, but not all limits are derivatives.

15. For $y(x)=5x^2$, find the limit of $y(x)$ as x approaches 3.

$$\lim_{x\to 3} y(x) = \lim_{x\to 3} 5x^2 = 5(3)^2 = 5\cdot 9 = 45$$

16. Using the Method of Increments, find the function for the instantaneous rate of change for $f(x) = 2x^3$.

$f(x) = 2x^3$



$$\frac{\Delta y}{\Delta x} = \frac{y(x+\Delta x) - y(x)}{\Delta x} = \frac{2(x+\Delta x)^3 - 2(x)^3}{\Delta x} = \frac{2(x+\Delta x)(x+\Delta x)(x+\Delta x) - 2x^3}{\Delta x}$$

$$= \frac{2(x^3 + 2x\Delta x + x(\Delta x)^2(x+\Delta x) - 2x^3}{\Delta x} =$$

$$= \frac{2(x^3 + 2x^2\Delta x + x(\Delta x)^2 + x^2\Delta x + 2x(\Delta x)^2 + (\Delta x)^3) - 2x^3}{\Delta x} =$$

$$= \frac{\cancel{2x^3} + \cancel{4x^2\Delta x}\ \cancel{2x(\Delta x)^2} + \cancel{2x^2\Delta x} + \cancel{4x(\Delta x)^2} + 2(\Delta x)^3 - \cancel{2x^3}}{\Delta x} =$$

$$= \frac{6x^2\Delta x + 6x(\Delta x)^2 + 2(\Delta x)^3}{\Delta x} = 6x^2 + 6x\Delta x + 2(\Delta x)^2$$

$f'(x)$ is the function that gives the instantaneous rate of change for $f(x) = 2x^3$, and so $f'(x) = \lim_{\Delta x\to 0}(6x^2 + 6x\Delta x + 2(\Delta x)^2) = 6x^2$.

17. Why is bx called a "first-order" term?

bx is called a first-order term because its highest degree is 1, that is, the highest exponent is 1. We could rewrite bx as bx^1.

18. How might you write $y(x) = c$ as a function of x?

$y(x) = c$ can be written as a function of x in this way: $y(x) = cx^0$. We remember that anything to the power of 0 is 1 and since anything multiplied by 1 is itself, we have two functions stated differently that mean the same thing—one with the x and one without.

19. In terms of our prime notation, how would you denote the derivative of $y(x) = f(x) + g(x) + u(x)$?

In terms of our prime notation, the derivative of $y(x) = f(x) + g(x) + u(x)$ would be $y'(x) = f'(x) + g'(x) + u'(x)$.

20. Using the Method of Increments, and approaching the value we're interested in, x, from values greater than x, find the derivative of $y(x) = bx$.

$$y(x) = bx$$

$$\frac{\Delta y}{\Delta x} = \frac{y(x+\Delta x) - y(x)}{\Delta x} = \frac{b(x+\Delta x) - bx}{\Delta x} = \frac{bx + b\Delta x - bx}{\Delta x} = \frac{b\Delta x}{\Delta x} = b$$

$$(\Delta x \neq 0)$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\left[\frac{\Delta y}{\Delta x}\right] = \lim_{\Delta x \to 0}(b) = b = \frac{dy}{dx}$$

21. Using the Power Rule, find the derivatives for the following power functions:

a. $y(x) = 4x^3$

Answer:

$$y(x) = 4x^3 \quad \Rightarrow \quad y' = 3 \cdot 4x^{3-1} = 12x^2$$

b. $V(r) = \frac{4}{3}\pi r^3$

Answer:

$$V(r) = \frac{4}{3}\pi r^3 \quad \Rightarrow \quad \frac{dV}{dr} = 3\left[\frac{4}{3}\right]\pi r^{3-1} \quad = 4\pi r^2$$

c. $y(x) = -16x^{-2} + 5x$

Answer:

$$y(x) = -16x^{-2} + 5x \quad \Rightarrow \quad y'(x) = -2\cdot(-16)x^{-2-1} + 1\cdot 5x^{1-1} =$$

$$= 32x^{-3} + 5 \quad = \frac{32}{x^3} + 5$$

d. $y(x) = 2x^{100}$

Answer:

$$y(x) = 2x^{100} \quad \Rightarrow \quad \frac{dy}{dx} = 100\cdot 2x^{100-1} \quad = 200x^{99}$$

e. $z(x) = \frac{1}{x}$

Answer:

$$z(x) = \frac{1}{x} = x^{-1} \quad \Rightarrow \quad z'(x) = -1\cdot x^{-1-1} \quad = -x^{-2} \quad = -\frac{1}{x^2}$$

f. $y(x) = \frac{1}{x} + 4x^{16} + 9x^9 + 1$

Answer:

$$y(x) = \frac{1}{x} + 4x^{16} + 9x^9 + 1 \quad = \quad -x^{-1} + 4x^{16} + 9x^9 + x^0$$

$$\Rightarrow \quad y'(x) = -1\cdot x^{-1-1} + 16\cdot 4x^{16-1} + 9\cdot 9x^{9-1} + 0\cdot x^{0-1} =$$

$$= -x^{-2} + 64x^{15} + 81x^8 = -\frac{1}{x^2} + 64x^{15} + 81x^8$$