CALCULUS FOR EVERYONE

EXAM 4 ANSWER KEY

COMPREHENSIVE FINAL EXAM

1. Where did we get the original Power Rule?

   We got the original Power Rule by using the Method of Increments.

2. Using the Reverse Power Rule, find the anti-derivatives of the following power functions. (You can always check your work by taking the derivative of your answer.)

   a. $y'(x) = 6x^3$

   Answer:

   $$y'(x) = 6x^3$$

   $$y(x) = \frac{6}{3+1}x^{3+1} + C = \frac{6}{4}x^4 + C = \frac{3}{2}x^4 + C$$

   b. $z(x) = \frac{1}{x}$

   Answer:

   $z(x) = \frac{1}{x} = x^{-1}$ We can't use the Reverse Power Rule for $n = -1$.

c. $f(x) = 9x^7 - \pi\sqrt[3]{x} + \dfrac{1}{\sqrt{x}} + x$

Answer:

$$f(x) = 9x^7 - \pi\sqrt[3]{x} + \frac{1}{\sqrt{x}} + x = 9x^7 - \pi x^{\frac{1}{3}} + x^{-\frac{1}{2}} + x$$

$$F(x) = \left[\frac{9}{7+1}x^{7+1} + C_1\right] - \left[\frac{\pi}{\frac{1}{3}+\frac{3}{3}}x^{\frac{1}{3}+\frac{3}{3}} + C_2\right] + \left[\frac{1}{-\frac{1}{2}+\frac{2}{2}}x^{-\frac{1}{2}+\frac{2}{2}} + C_3\right] +$$
$$+\left[\frac{1}{1+1}x^{1+1} + C_4\right] =$$

NOTE: The negative sign in front of the second parentheses gets distributed, but a constant minus a constant is still a constant.

$$= \frac{9}{8}x^8 - \frac{\pi}{\frac{4}{3}}x^{\frac{4}{3}} + \frac{1}{\frac{3}{2}}x^{\frac{3}{2}} + \frac{1}{\frac{1}{2}}x^{\frac{1}{2}} + C =$$

$$F(x) = \frac{9}{8}x^8 - \frac{3\pi}{4}x^{\frac{4}{3}} + \frac{2}{3}x^{\frac{3}{2}} + 2\sqrt{x} + C$$

3. What is the derivative of

$f(x) = \dfrac{a}{n+1}x^{n+1} + C$

Answer:

$$f(x) = \frac{a}{n+1}x^{n+1} + C \qquad = \frac{a}{n+1}x^{n+1} + Cx^0$$

$$f'(x) = (n+1)\frac{a}{(n+1)}x^{(n+1)-1} + 0 \cdot C\,x^{0-1} =$$

$$f'(x) = a\,x^n$$

4. Using the Reverse Power Rule, find the height, velocity, and acceleration functions for the following situations:

   a. An object dropped from a height of 100 feet.

   We begin with $a(t) = -32$ for any freefall case. To find $v(t)$ we note that $v(0) = 0$ because the object was dropped.

   $$v(t) = A(t) = \frac{-32}{0+1}t^{0+1} + C = -32t + C$$
   $$v_0 = v(0) = 0 = -32(0) + C \quad \Rightarrow \quad C = 0$$

   Therefore,

   $$v(t) = -32t$$

   Now, we know that $h(0) = 100$, and so

   $$h(t) = v(t) = \frac{-32}{1+1}t^{1+1} + C = \frac{-32}{2}t^2 + C = -16t^2 + C$$
   $$h(0) = 100 = -16(0)^2 + C \quad \Rightarrow \quad C = 100$$

   Therefore,

   $$h(t) = -16t^2 + 100$$

   b. An object thrown straight up in the air at a speed of 25 ft/s from a height of 50 ft.

   $a(t) = -32$, which is the same for all free fall cases.

   $$v(t) = \frac{-32}{0+1}t^{0\ \ 1} + C = -32t + C$$
   $$v(0) = 25 = -32(0) + C \quad \Rightarrow \quad C = 25$$

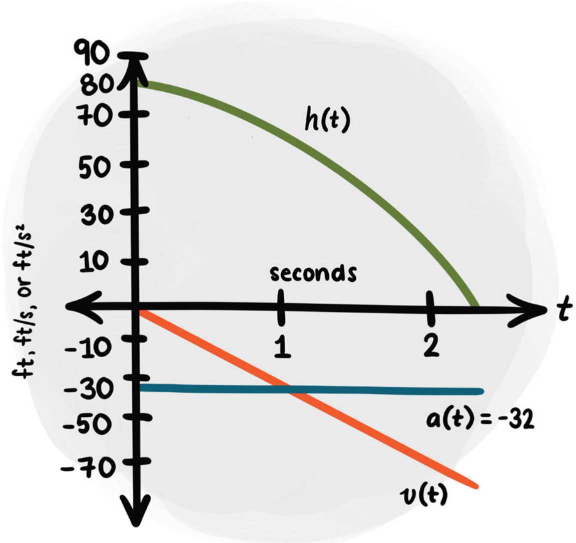
   Therefore,

   $$v(t) = -32t + 25$$
   $$h(t) = V(t) = \frac{-32}{1+1}t^{1+1} + C_1 + \frac{25t}{0+1}t^{0+1} + C_2 = -16t^2 + 25t + C$$
   $$h(0) = 50 = -16(0)^2 + 25(0) + C \quad \Rightarrow \quad C = 50$$

   Therefore,

   $$h(t) = -16t^2 + 25t + 50$$

c. An object in free fall whose behavior is described by the following graph:



As usual $a(t) = -32$

$$v(t) = A(t) = \frac{-32}{0+1}t^{0+1} + C_1 = -32t + C_1$$

From the graph we see that at $t = 0$, $v(t) = 0$ and so

$$v(0) = 0 = -32(0) + C_1 \quad \Rightarrow \quad C_1 = 0$$

Therefore,

$$v(t) = -32t$$

$$h(t) = \frac{-32}{1+1}t^{1+1} + C_2 = -16t^2 + C_2$$

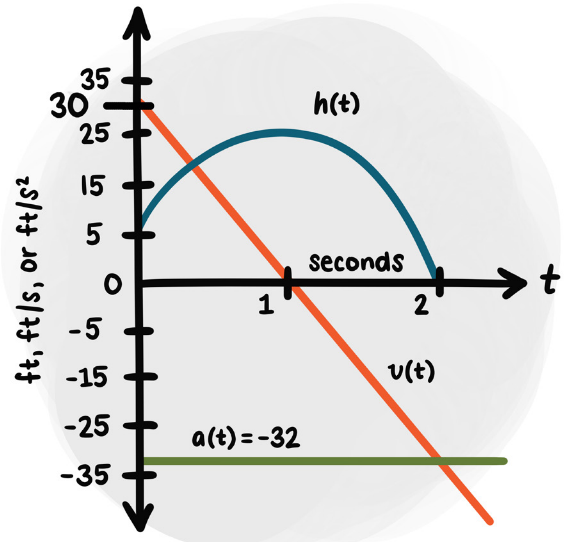
From the graph we see that at $t = 0$, $h(t) = 80$ and so

$$h(0) = 80 = -16(0)^2 + C_2 \quad \Rightarrow \quad C_2 = 80$$

Therefore,

$$h(t) = -16t^2 + 80$$

d. An object in free fall whose behavior is described by the following graph:



$a(t) = -32$

$v(t) = A(t) = \dfrac{-32}{0+1}t^{0+1} + C_1 = -32t + C_1$

From the graph we see that at $t = 0$, $v\left(t\right) = 30$ and so

$v(0) = 30 = -32(0) + C_1 \quad \Rightarrow \quad C_1 = 30$

Therefore,

$v(t) = -32t + 30$

$h(t) = V(t) = \dfrac{-32}{1+1}t^{1+1}C_2 + \dfrac{30}{0+1}t^{0+1} + C_3$

$= -16t^2 + 30t + C$

From the graph we see that at $t = 0$, $h\left(t\right) = 5$ and so

$h(0) = 5 = -16(0)^2 + 30(0) + C \quad \Rightarrow \quad C = 5$

Therefore,

$h(t) = -16t^2 + 30t + 5$

5. What do we call the case of overestimating? Of underestimating?

   In the case of overestimating, we say "circumscribed," and in the case of underestimating, we say "inscribed."

6. In terms of S_n, and the limit of notation, give the definition of the integral.

   In terms of S_n, and the limit of notation, the definition of the integral is $A = \lim_{n \to \infty} S_n$.

7. In terms of A_i's and the limit notation, give the definition of the integral.

   In terms of A_i's and the limit notation, the definition of the integral is $A = \lim_{n \to \infty} \left[A_1 + A_2 + A_3 + A_4 + \cdots + A_n \right]$.

8. In terms of y's and Δx's and the limit notation, give the definition of the integral.

   In terms of y's and Δx's and the limit notation, the definition of the integral is $A = \lim_{n \to \infty} \left[(y_1 \cdot \Delta x) + (y_2 \cdot \Delta x) + (y_3 \cdot \Delta x) + (y_4 \cdot \Delta x) + \cdots + (y_n \cdot \Delta x) \right]$.

9. What is

$$\lim_{n \to \infty} S_n$$

   in terms of our elongated "s" integral notation? What does "*ydx*" as a whole stand for? What does y stand for? What does dx stand for?

   In terms of our elongated "s" integral notation, $\lim_{x \to \infty} S_n$ *is* $\int_a^b ydx$. As a whole, "*ydx*" stands for the area of each infinitely thin mini-rectangle. *y* stands for the height of each mini-rectangle, and *dx* stands for the base of each mini-rectangle.

10. What do a and b stand for in the following formula?

$$\int_a^b y\,dx$$

In the formula $\int_a^b y\,dx$, the a and b stand for the x-values of the endpoints of the entire area's base.

11. What does "$f(x)dx$" as a whole stand for in the following formula?

$$\int_a^b f(x)dx$$

What does $f(x)$ stand for? What does dx stand for?

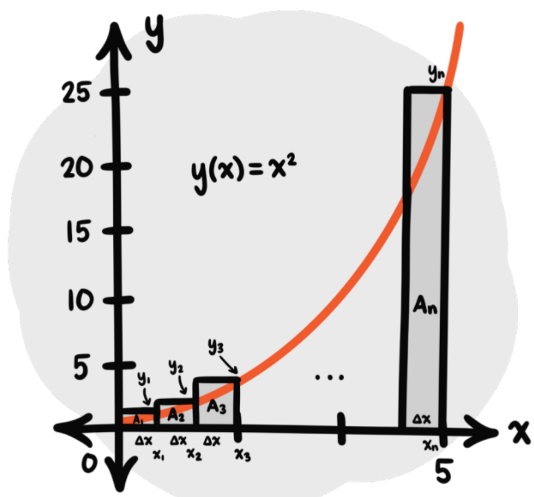
As a whole, "$f(x)dx$" stands for the area of each infinitely thin mini-rectangle. $f(x)$ stands for the height of each mini-rectangle, and dx stands for the base of each mini-rectangle.

12. Find

$$\int_0^5 x^2dx$$

beginning with an *overestimation* of the sum of the mini-rectangles.

Integral of $y = x^2$ from $x = 0$ to $x = 5$.



i	x_i	y_i
1	$x_1 = \Delta x$	$y_1 = (\Delta x)^2$
2	$x_2 = 2\Delta x$	$y_2 = (2\Delta x)^2 = 4(\Delta x)^2$
3	$x_3 = 3\Delta x$	$y_3 = (3\Delta x)^2 = 9(\Delta x)^2$
$\vdots$	$\vdots$	$\vdots$
n	$x_n = n\Delta x$	$y_n = (n\Delta x)^2 = n^2(\Delta x)^2$

$$\begin{aligned}\overline{S}_n &= (\Delta x)^2\Delta x + 4(\Delta x)^2\Delta x + 9(\Delta x)^2\Delta x + \cdots + n^2(\Delta x)^2\Delta x = \\ &= (\Delta x)^3 + 4(\Delta x)^3 + 9(\Delta x)^3 + \cdots + n^2(\Delta x)^3 = \\ &= (\Delta x)^3\underbrace{\left[1^2 + 2^2 + 3^2 + \cdots + n^2\right]}_{\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}} = \\ &= (\Delta x)^3\left[\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right]\end{aligned}$$

We need Δx in terms of n:

$$\Delta x = \frac{5-0}{n} = \frac{5}{n}$$

$$\begin{aligned}\underline{S}_n &= \left[\frac{5}{n}\right]^3\left[\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right] = \frac{125}{n^3}\left[\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right] = \\ &= 125\left[\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right] = \frac{125}{3} + \frac{125}{2n} + \frac{125}{6n^2}\end{aligned}$$

$$\Rightarrow \quad A = \int_0^5 x^2 dx = \lim_{n\to\infty}\left[\frac{125}{3} + \frac{125}{2n} + \frac{125}{6n^2}\right] = \frac{125}{3}$$

13. What are the three central concepts in calculus? What ties them all together?

    The three central concepts in calculus are the limit, derivative, and integral of a function. The Fundamental Theorem of Calculus ties them all together.

14. In terms of limits, what is the formula for the definition of a derivative?

The formula for the definition of a derivative is $\frac{dy}{dx} = \lim\limits_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right)$.

15. In terms of limits, what is the formula for the definition of an integral?

The formula for the definition of an integral is $\int_a^b y(x)dx = \lim\limits_{n \to \infty} S_n$.

16. Calculuate the following definite integral:

$\int_1^3 6x^3 dx$

Answer:

$$\int_1^3 6x^3 dx = \frac{6}{4}x^4 + C\Big|_1^3 = \left[\frac{3}{2}(3)^4 + C\right] - \left[\frac{3}{2}(1)^4 + C\right] =$$

$$= \frac{3(81)}{2} + C - \frac{3}{2} - C = \frac{243}{2} - \frac{3}{2} = \frac{240}{2} = 120$$

17. Find the following indefinite integrals:

a. $\int x^2 dx$

Answer:

$$\int x^2 dx = \frac{1}{3}x^3 + C$$

b. $\int\left(-\frac{3\sqrt{x}}{7}\right)dx$

Answer:

$$\int\left[-\frac{3\sqrt{x}}{7}\right]dx = \int\left[-\frac{3}{7}x^{\frac{1}{2}}\right]dx = -\frac{3}{7}\cdot\frac{1}{\left[\frac{1}{2}+\frac{2}{2}\right]}x^{\frac{1}{2}+\frac{2}{3}} + C =$$

$$= -\frac{3}{7\cdot\frac{3}{2}}x^{\frac{3}{2}} + C = -\frac{2}{7}x^{\frac{3}{2}} + C = -\frac{2}{7}\sqrt{x^3} + C$$

18. Using the Fundamental Theorem of Calculus, find the height, velocity, and acceleration functions for an object dropped from a height of 100 feet.

We know that for *any* free fall scenario that $a(t) = -32$ and that $v(t) = A(t) = \int(-32)\,dt = -32t + C.$
In *this* particular case, we're told that $v(0) = 0$ (because the object is simply dropped) and so

$$v_0 = v(0) = 0 = -32(0) + C \quad \Rightarrow \quad C = 0$$

This gives us

$$v(t) = -32t.$$

Now,

$$h(t) = v(t) = \int -32t\,dt = -\frac{32}{2}t^2 + C = -16t^2 + C,$$

where,

$$h(0) = 100 = -16(0)^2 + C \quad \Rightarrow \quad C = 100$$

and so

$$h(t) = -16t^2 + 100.$$

19. What geometrical property is the derivative associated with? What physical quantities can this geometrical property be associated with?

Geometrically, the derivative is associated with the slope of a function. This geometrical property, slope, can be associated with the physical properties of speed and acceleration.

20. What geometrical property is the integral associated with? What physical quantities can this geometrical property be associated with? Why didn't we talk about this earlier when introducing integrals?

Geometrically, the integral is associated with areas, specifically, the areas under a function's curve. This geometrical property, area, can be associated with the physical properties of "change in velocity" and "change in height." We didn't talk about the physical association of this geometrical property earlier because we didn't have a relatively simple method to calculate integrals, not until we explored the Fundamental Theorem of Calculus.

21. What physical property is represented by the area "under" the acceleration curve?

The physical property represented by the area "under" the acceleration curve is "change in velocity."

22. What physical property is represented by the area "under" the velocity curve?

The physical property represented by the area "under" the velocity curve is "change in distance" or "change in height" in the case of free fall.

23. Find the change in height and velocity between a seconds and b seconds for a free fall situation described by the following formula:

$$h(t) = -16t^2 + v_0 t + h_0$$

This exercise is asking you to make a very general calculation, one that represents any beginning time and any ending time. As usual,

$$h(t) = -16t^2 + v_0 t + h_0 \quad \Rightarrow \quad v(t) = -32t + v_0 \quad \Rightarrow \quad a(t) = -32$$

The change in speed, Δv, is given by the area "under" the acceleration curve (let's again call it A_a, where the subscript a stands for "acceleration" and not for the beginning of time, which we also called a).

$$\Delta v = A_a = \int_a^b -32\,dt = -32t\Big| + C_a^b = -32(b) + C - \left[-32(a) + C\right] =$$
$$= -32 \cdot b + 32 \cdot a = -32(b - a)$$

The change in the height, Δh, is given by the area "under" the velocity curve (let's call is A_v):

$$\begin{aligned}\Delta h = A_v &= \int_a^b (-32t + v_0)\,dt = -16t^2 + v_0 t + C\Big|_a^b = \\ &= -16(b)^2 + v_0(b) + C - \left[-16(a)^2 + v_0(a) + C\right] = \\ &= -16b^2 + v_0 b + 16a^2 - v_0 a = \\ &= -16(b^2 - a^2) + v_0(b - a)\end{aligned}$$

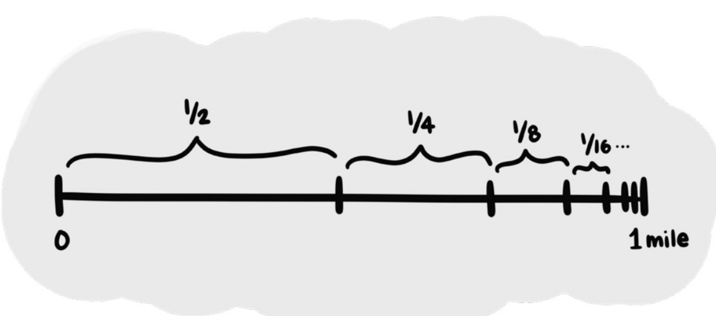
24. What were the three Pythagorean ideas that were at the foundation of the Scientific Revolution?

The three Pythagorean ideas that were at the foundation of the Scientific Revolution were that the universe is ordered according to perfect mathematical laws, that divine reason is the orderer, and that human reason can discern the divine mathematical pattern.

25. Using calculus, solve Zeno's paradox.

Zeno's paradox can be mathematically formulated by noting that the progressive dividing in half of the total distance (say, one mile) can be written as follows:

$$\text{Sum} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$



We can write the above sum in terms of *n* number of divisions (let's call this definite sum, Sum_n):

$$\begin{aligned}\text{Sum}_n &= \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots+\frac{1}{2^n}= \\ &= \frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\cdots+\frac{1}{2^n}\end{aligned}$$

To get rid of the "…" we note the simple and obvious fact (though it's not obvious at first *why* we would do this) that

$$\text{Sum}_n = \frac{1}{2}\text{Sum}_n + \frac{1}{2}\text{Sum}_n .$$

This can be rearranged as follows (though again, it might not seem to be entirely clear at this stage why we would do this):

$$\text{Sum}_n - \frac{1}{2}\text{Sum}_n = \frac{1}{2}\text{Sum}_n .$$

Let's now write this subtraction problem vertically:

$$\begin{array}{l} \text{Sum}_n \\ -\frac{1}{2}\text{Sum}_n \\ \hline \frac{1}{2}\text{Sum}_n \end{array}$$

And $\frac{1}{2}\text{Sum}_n$ is equal to

$$\begin{aligned}\frac{1}{2}\text{Sum}_n &= \frac{1}{2}\cdot\left[\frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\cdots+\frac{1}{2^n}\right] \\ &= \frac{1}{2^1\cdot 2^1}+\frac{1}{2^1\cdot 2^2}+\frac{1}{2^1\cdot 2^3}+\frac{1}{2^1\cdot 2^4}+\cdots+\frac{1}{2^1\cdot 2^n} \\ &= \frac{1}{2^{1+1}}+\frac{1}{2^{2+1}}+\frac{1}{2^{3+1}}+\frac{1}{2^{4+1}}+\cdots+\frac{1}{2^{n+1}} \\ &= \frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\frac{1}{2^5}+\cdots+\frac{1}{2^{n+1}}\end{aligned}$$

Plugging this result into the vertical subtraction problem:

$$\begin{array}{rl} \text{Sum}_n &= \frac{1}{2^1} + \cancel{\frac{1}{2^2}} + \cancel{\frac{1}{2^3}} + \cancel{\frac{1}{2^4}} + \cancel{\frac{1}{2^5}} + \cdots + \cancel{\frac{1}{2^n}} \\ -\frac{1}{2}\text{Sum}_n &= \cancel{\frac{1}{2^2}} + \cancel{\frac{1}{2^3}} + \cancel{\frac{1}{2^4}} + \cancel{\frac{1}{2^5}} + \cdots \cancel{\frac{1}{2^n}} + \frac{1}{2^{n+1}} \\ \hline \frac{1}{2}\text{Sum}_n &= \frac{1}{2} - \frac{1}{2^{n+1}} \end{array}$$

Taking this answer to the subtraction problem, solve for Sum_n (the algebra is carried out in a slightly different order than in the chapter but it's entirely equivalent):

$$\begin{aligned} \text{Sum}_n &= 2\left[\frac{1}{2} - \frac{1}{2^{n+1}}\right] \\ &= \frac{2}{2} - \frac{2}{2^{n+1}} \\ &= 1 - \frac{1}{2^{n+1-1}} \\ &= 1 - \frac{1}{2^n} \end{aligned}$$

The total distance, then, is

$$\lim_{n\to\infty}\left[1 - \frac{1}{2^n}\right] = 1\,\text{mile}$$

And so the total distance is traversed.